Microstrip Propagation on Magnetic Substrates—Part I: Design Theory

ROBERT A. PUCEL, SENIOR MEMBER, IEEE, AND DANIEL J. MASSÉ, MEMBER, IEEE

Abstract—**Formulas and graphs are presented for the effective relative permeability and the filling factors of magnetic substrates in microstrip. Both the propagation and the magnetic loss filling factors are included. In the calculation of these quantities, use was made of the filling factors for dielectric substrates obtained from Wheeler's analysis and a duality relationship between magnetic and dielectric substrates derived in this paper.**

I. Introduction

A LARGE BODY of design information for microstrip on dielectric substrates has been accumulated over the last few years [1]–[3]. Equivalent design data for magnetic substrates are incomplete. It is our purpose to present the missing data in a form most useful to the design engineer. Before proceeding, we shall review briefly some basic formulas for dielectric substrates.

A cross section of microstrip on a dielectric–magnetic substrate is shown in Fig. 1. Provided the frequency is not too high, this structure will propagate a wave which for all practical purposes is a transverse electromagnetic wave. If the dielectric constant k of the substrate is much greater than unity, most of the electric energy is confined to the dielectric region in the vicinity of the strip conductor and ground plane. However, because some of the electric field also fringes out into the air space above the strip conductor, the value of the effective dielectric constant k_{eff} entering into the calculation of the characteristic impedance and phase velocity is less than k, that is $1 < k_{\text{eff}} < k$. In other words, the propagation "filling factor" for the dielectric, here denoted as q_d, and defined by Wheeler [1] as

$$q_d = \frac{k_{\text{eff}} - 1}{k - 1} \tag{1}$$

is less than unity. Both k_{eff} and q_d are functions of the dielectric constant k and the geometrical factor w/h, the ratio of the conductor strip width to substrate height. This functional dependence can be derived from Wheeler's paper.

If dielectric losses are present, the effective value of the dielectric loss tangent $\tan \delta_{\text{eff}}$ is also less than the loss tangent of the substrate $\tan \delta_d$ and can be expressed in the form [4], [5]

Manuscript received May 27, 1971; revised August 12, 1971.
The authors are with the Research Division, Raytheon Company, Waltham, Mass. 02154.

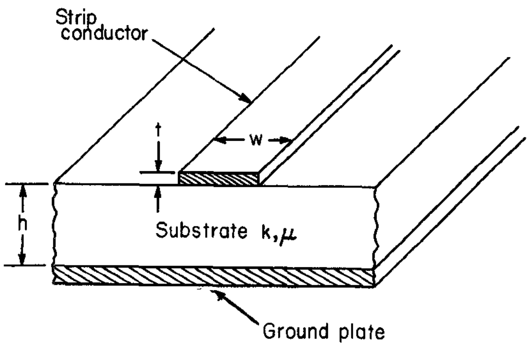

Fig. 1. Microstrip.

$$\tan \delta_{d_{\text{eff}}} = q_{d,l} \tan \delta_d \tag{2}$$

where $q_{d,l}$ is a filling factor for losses given by [5]

$$q_{d,l} = q_d \frac{k}{k_{\text{eff}}} = \frac{1 - k_{\text{eff}}^{-1}}{1 - k^{-1}}. \tag{3}$$

II. Magnetic Substrates

It would be convenient to have equivalent design formulas for substrates with magnetic properties. Fortunately, this information can be obtained from the above expressions by using a duality relationship for dielectric and magnetic substrates developed in the Appendix. This duality, based on an observation of Kaneki [6], allows one to calculate the functional dependence of the effective relative permeability μ_{eff} on w/h and the relative permeability μ of the substrate, once the functional dependence of k_{eff} on w/h and k is known. Thus the solution for the magnetic field distribution can be bypassed.

The duality relationship (which derives from the duality of k and $1/\mu$ in Maxwell's equations) is based on a TEM-mode approximation for the magnetic case, the same assumption as used for the dielectric case [1]–[3]. This relationship takes the form

$$\mu_{\text{eff}}(w/h, \mu) = \frac{1}{k_{\text{eff}}(w/h, \mu^{-1})}. \tag{4}$$

Note that the duality amounts to the conversions $k \rightarrow 1/\mu$ and $k_{\text{eff}} \rightarrow 1/\mu_{\text{eff}}$ in the formulas for the dielectric case. Equation (4) implies that one need not make a separate determination of the effective relative permeability if one has at hand tables or graphs of the effective dielectric constant.

It follows from (4) and (3) that a magnetic filling factor for propagation can be defined as

$$q_m = \frac{\mu_{\rm eff}^{-1} - 1}{\mu^{-1} - 1}. \tag{5}$$

Note that $q_m(w/h, \mu) = q_d(w/h, \mu^{-1})$.

In like manner the expressions for the filling factor of the magnetic loss tangent tan δ_m and the effective value of this loss tangent take the form

$$q_{m,l} = q_m \frac{\mu_{\rm eff}}{\mu} = \frac{1 - \mu_{\rm eff}}{1 - \mu} \tag{6}$$

or

$$\tan \delta_{m\rm eff} = q_{m,l} \tan \delta_m. \tag{7}$$

Our TEM assumption allows us to write simple formulas for the characteristic impedance Z_0, guide wavelength λ_g, and total substrate loss per wavelength $\alpha\lambda_g$ for microstrip on a substrate exhibiting both dielectric and magnetic properties. Thus we have

$$Z_0 = Z_0' \sqrt{\frac{\mu_{\rm eff}}{k_{\rm eff}}} \quad (\Omega) \tag{8}$$

$$\lambda_g = \lambda_0 / \sqrt{k_{\rm eff}\mu_{\rm eff}} \quad ({\rm cm}) \tag{9}$$

$$\alpha\lambda_g = 27.3(\tan \delta_{d\rm eff} + \tan \delta_{m\rm eff}) \quad ({\rm dB}). \tag{10}$$

The wavelength λ_0 corresponds to free space. Here Z_0' is the characteristic impedance when $\mu = k = 1$, which can be calculated exactly from the capacitance per unit length [7] for an air dielectric or from Wheeler's expressions letting $k = 1$ [1]. The attenuation (10) of course only represents the substrate losses, to which must be added the contribution from conductor losses [5].

In Section III we shall present explicit formulas for $\mu_{\rm eff}$ and, by way of review, for $k_{\rm eff}$ based on Wheeler's formulas.

III. Derivation of Formulas

Wheeler's analytic expressions [1] for the characteristic impedance of microstrip as a function of dielectric constant and the ratio w/h are given in piecewise form, one solution valid for $w/h \leq 2$, the other for $w/h \geq 2$. The two solutions do not join exactly at $w/h = 2$ (about a 5–10-percent error). His results may be graphically smoothed in this vicinity to join properly.

If we take Wheeler's expressions for Z_0, and set $k = 1$ to give Z_0', the value for an air dielectric, then the effective value of dielectric constant may be obtained from

$$k_{\rm eff} = \left[\frac{Z_0(w/h, 1)}{Z_0(w/h, k)}\right]^2. \tag{11}$$

For $w/h \leq 2$

$$k_{\rm eff} = \frac{1 + k}{2}\left(\frac{A}{A - B}\right)^2 \tag{12a}$$

and for $w/h \geq 2$

$$k_{\rm eff} = k\left(\frac{C - D}{C}\right)^2 \tag{12b}$$

where

$$A = \ln \frac{8h}{w} + \frac{1}{32}\left(\frac{w}{h}\right)^2 \tag{13a}$$

$$B = \frac{1}{2}\left(\frac{k - 1}{k + 1}\right)\left[\ln \frac{\pi}{2} + \frac{1}{k} \ln \frac{4}{\pi}\right] \tag{13b}$$

$$C = \frac{w}{2h} + \frac{1}{\pi}\left[\ln 2\pi\epsilon \left(\frac{w}{2h} + 0.94\right)\right] \tag{13c}$$

$$D = \frac{k - 1}{2\pi k}\left\{\ln \left[\frac{\pi\epsilon}{2}\left(\frac{w}{2h} + 0.94\right)\right] - \frac{1}{k} \ln \left(\frac{\epsilon\pi^2}{16}\right)\right\} \tag{13d}$$

from which the dielectric filling factors q_d and $q_{d,l}$, (1) and (3), respectively, may be computed.

The expressions for the effective relative permeability may be derived from the above by employing the duality relationship (4). Thus for $w/h \leq 2$

$$\mu_{\rm eff} = \frac{2\mu}{1 + \mu}\left(\frac{A - B'}{A}\right)^2 \tag{14a}$$

and for $w/h \geq 2$

$$\mu_{\rm eff} = \mu\left(\frac{C}{C - D'}\right)^2 \tag{14b}$$

where A and C are given by (13) and B' and D' are derived from B and D by letting $k \rightarrow \mu^{-1}$, that is,

$$B' = \frac{1}{2}\left(\frac{1 - \mu}{1 + \mu}\right)\left[\ln \frac{\pi}{2} + \mu \ln \frac{4}{\pi}\right] \tag{15a}$$

$$D' = \frac{1 - \mu}{2}\left\{\ln \left[\frac{\pi\epsilon}{2}\left(\frac{w}{2h} + 0.94\right)\right] - \mu \ln \left(\frac{\epsilon\pi^2}{16}\right)\right\}. \tag{15b}$$

The two filling factors q_m and $q_{m,l}$, (5) and (6), respectively, may be calculated by use of the expressions for $\mu_{\rm eff}$ above.

Equations (14) and (15) together with (5)–(7) provide all the information necessary to design microstrip on magnetic substrates.

IV. Graphical Results

Since the purpose of this paper is to present design graphs for microstrip on ferrite and garnet substrates, our computations for $\mu_{\rm eff}$, q_m, and $q_{m,l}$ were made for values of magnetic constant less than unity, and indeed only for the practical range $0.4 < \mu < 1$. Because μ is less

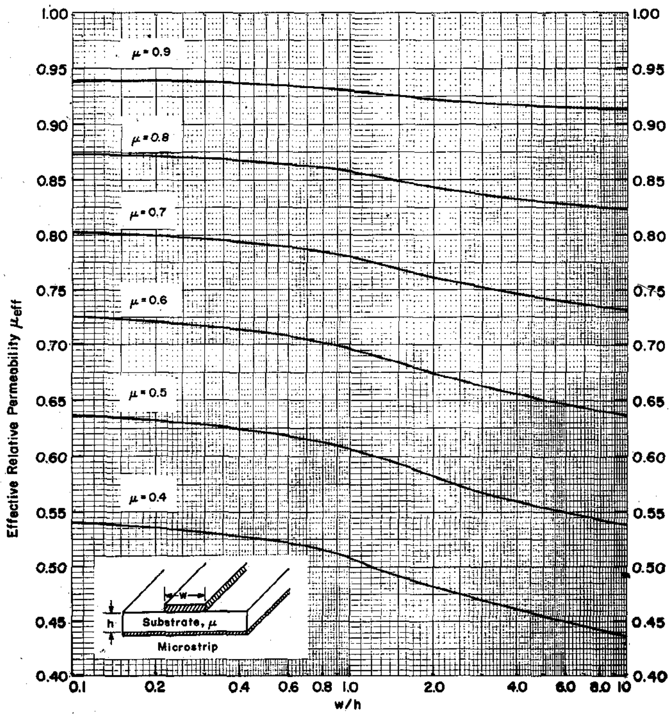

Fig. 2. Effective relative permeability of microstrip as a function of the relative permeability of the substrate and the geometrical parameter w/h.

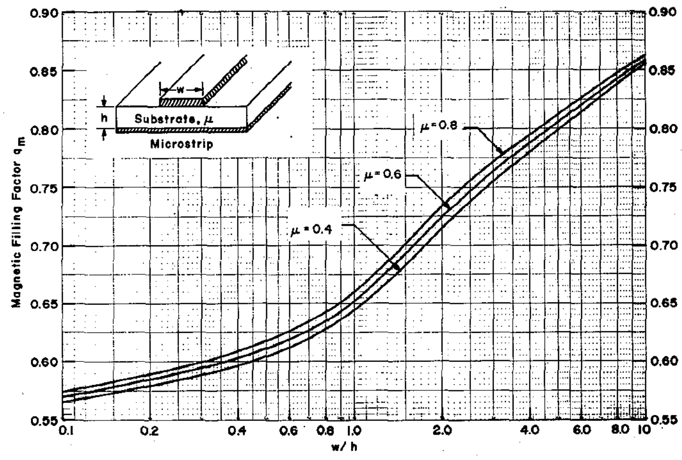

Fig. 3. Propagation magnetic filling factor of microstrip as a function of the relative permeability of the substrate and the geometrical parameter w/h.

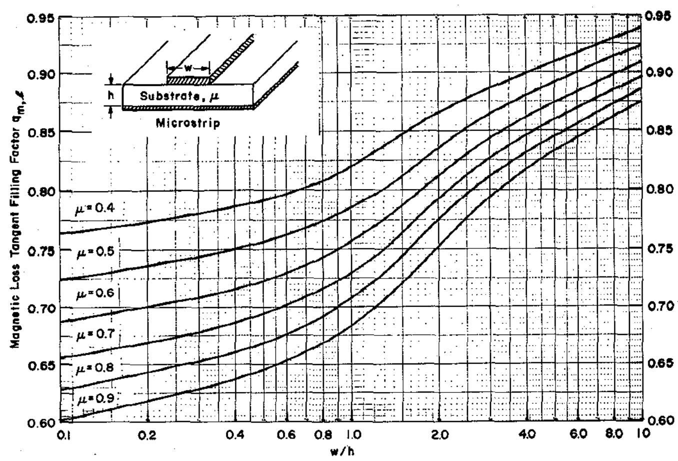

Fig. 4. Filling factor for magnetic loss tangent of microstrip as a function of the relative permeability of the substrate and the geometrical parameter w/h.

than unity, the value for the air space above the microstrip, one should expect the effective value μ_{eff} to fall between μ and unity, that is $\mu < \mu_{\text{eff}} < 1$. One can show that for $w/h \to 0$, $\mu_{\text{eff}}^{-1} \to \frac{1}{2}(1+\mu^{-1})$, and for $w/h \to \infty$, $\mu_{\text{eff}} \to \mu$.[1] In other words, μ_{eff} is bracketed in the range

$$\mu < \mu_{\text{eff}} < \frac{2\mu}{1+\mu} \cdot \tag{16}$$

The curves of μ_{eff} in Fig. 2 illustrate this expected behavior.

The filling factors q_m and $q_{m,l}$ derived from μ are shown in Figs. 3 and 4. Because of the mild dependence of q_m on μ, only three curves were plotted to avoid crowding.

Observe in Fig. 4 that the loss filling factor becomes larger with increasing w/h, a reflection of the growing importance of the substrate. Using the limits on μ_{eff} derived above, one may show that $q_{m,l}$ falls in the range

$$\frac{1}{1+\mu} < q_{m,l} < 1. \tag{17}$$

Experimental verification of our design formulas are given in Part II of this paper [8], where we apply them to ferrite and garnet substrates, which, operated in certain biasing states, can be treated to a good approximation as reciprocal media.

[1] Wheeler shows that $\frac{1}{2}(k+1) < k_{\text{eff}} < k$, the lower limit applying to $w/h \to 0$, the upper to $w/h \to \infty$. Our results for the magnetic case are derived with the help of (4).

V. Summary

Design formulas and graphs were presented for the effective relative permeability and the propagation and attenuation filling factors of microstrip on magnetic substrates. The formulas were obtained by application of a duality relationship which exists between magnetic and dielectric substrates which circumvents the need for solution of the magnetic field distribution in microstrip.

Appendix

We wish to justify the duality relationship between the effective values of the dielectric constant and the relative permeability cited in (4), and establish the conditions under which it is valid.

Our point of departure is an enumeration of the assumptions for our analysis, namely, 1) TEM mode of propagation, 2) perfect conductors (infinite conductivity), and 3) isotropic, homogeneous, nongyromagnetic

TABLE I

RELATIONS PERTAINING TO SOLUTION FOR ELECTRIC AND MAGNETIC FIELDS

Condition	Electric Potential	Magnetic Potential
Potential function	$\psi_e(x, y, k)$	$\boldsymbol{k}\psi_m(x, y, \mu)$
Field vectors	$\boldsymbol{E} = -\nabla\psi_e$	$\boldsymbol{B} = \nabla \times (\boldsymbol{k}\psi_m) = -\boldsymbol{k} \times \nabla\psi_m$
	$\boldsymbol{D} = \epsilon_0 k(x, y)\boldsymbol{E}$	$\boldsymbol{H} = \mu_0^{-1}\mu^{-1}(x, y)\boldsymbol{B}$
Laplace's equation	$\nabla^2\psi_e = 0$	$\nabla^2\psi_m = 0$
Boundary conditions		
on conductor surfaces S_i, S_0	$k(x, y)\nabla\psi_e \cdot \boldsymbol{n} = -\sigma(s)/\epsilon_0$	$(\boldsymbol{k} \times \boldsymbol{n}) \cdot \nabla\psi_m = 0$
	$(\boldsymbol{k} \times \boldsymbol{n}) \cdot \nabla\psi_e = 0$	$\mu^{-1}(x, y)\nabla\psi_m \cdot \boldsymbol{n} = -\mu_0 j(s)$
on interface S_{ai}	$k\nabla\psi_{e,1} \cdot \boldsymbol{n} = \nabla\psi_{e,2} \cdot \boldsymbol{n}$	$(\boldsymbol{k} \times \boldsymbol{n}) \cdot \nabla\psi_{m,1} = (\boldsymbol{k} \times \boldsymbol{n}) \cdot \nabla\psi_{m,2}$
	$(\boldsymbol{k} \times \boldsymbol{n}) \cdot \nabla\psi_{e,1} = (\boldsymbol{k} \times \boldsymbol{n}) \cdot \nabla\psi_{e,2}$	$\mu^{-1}\nabla\psi_{m,1} \cdot \boldsymbol{n} = \nabla\psi_{m,2} \cdot \boldsymbol{n}$

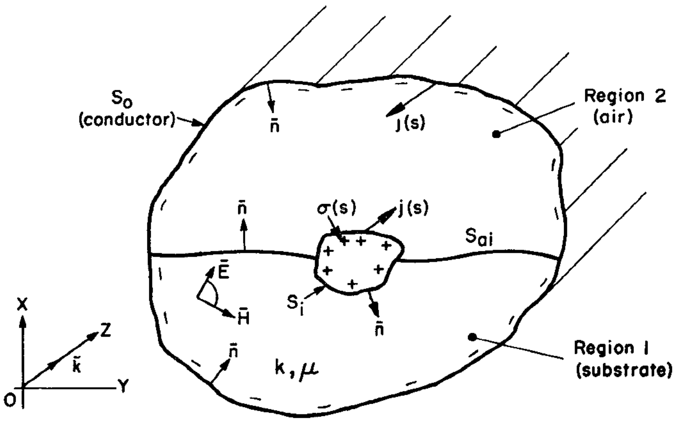

Fig. 5. Cross section of TEM structure relevant to derivation of duality relationship.

substrate. These assumptions are the ones used in all microstrip analyses; hence they are not restrictive for our purposes [1]–[3].

We assume for generality a cylindrical system of arbitrary cross section whose axis is along the Z direction, the assumed direction of propagation as illustrated in Fig. 5. Although a closed system is shown for convenience, our analysis also applies to open systems as well. The assumption of a closed outer conductor is not restrictive, since practical microstrip structures usually have an enclosure for shielding purposes.

With perfect conductors, the currents and charges, denoted by the surface densities $j(s)$ and $\sigma(s)$, reside on the conductor surfaces as shown in Fig. 5. Here s is a transverse surface coordinate on the conductors. For a TEM mode, the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{B} = \mu\boldsymbol{H}$ are in the x–y (transverse) plane. Their spatial distributions in this plane are solutions of a two-dimensional Laplace equation.

It is convenient to express $\boldsymbol{E}$ as the gradient of a scalar potential function ψ_e and $\boldsymbol{B}$ as the curl of a *vector* potential function $\boldsymbol{A}$ which is directed along the Z axis, the direction of the current; that is, $\boldsymbol{A}(x, y) = \boldsymbol{k}\psi_m$, where $\boldsymbol{k}$ is a unit vector along the Z axis. Thus

$$\boldsymbol{E} = -\nabla\psi_e(x, y, k) \tag{18}$$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = -\boldsymbol{k} \times \nabla\psi_m(x, y, \mu). \tag{19}$$

Note that like $\boldsymbol{E}$, $\boldsymbol{B}$ is also proportional to the gradient of a scalar function (not to be confused with a scalar magnetic potential). Observe that ψ_e depends on the dielectric constant of the substrate k, but not on the relative permeability μ of the substrate. The converse is true for ψ_m. This is characteristic of a TEM solution. Since Laplace's equation is satisfied by the vector and magnetic potentials, then $\nabla^2\psi_e = 0$ and $\nabla^2\boldsymbol{A} = \boldsymbol{k}\nabla^2\psi_m = 0$ or $\nabla^2\psi_m = 0$. The solutions of these equations are determined by the geometry and the usual boundary conditions imposed on $\boldsymbol{E}$ and $\boldsymbol{B}$ at the conductor surfaces and at the substrate–air interface. These are summarized in Table I.

Perusal of Table I shows that ψ_e and ψ_m satisfy identical *sets* of boundary conditions provided the normal and tangential boundary conditions for $\boldsymbol{E}$ and $\boldsymbol{B}$ are interchanged (which is of no consequence to the solutions for ψ_e and ψ_m). Thus the *form* of the solution for ψ_m is identical to the *form* of the solution for ψ_e, if k is replaced by μ^{-1} and provided the surface densities $\sigma(s)$ and $j(s)$ are proportional. Assuming for a moment the latter to be true, then because of the linearity of the system, we may express ψ_e and ψ_m in the form

$$\psi_e(x, y, k) = \epsilon_0^{-1}QF(x, y, k) \tag{20}$$

$$\psi_m(x, y, \mu) = \mu_0 IF(x, y, \mu^{-1}) \tag{21}$$

where F is a scalar function satisfying Laplace's equation and $Q = \oint_{S_i}\sigma(s)\, ds$, $I = \int_{S_i} j(s)\, ds$ are the total charge/length and current on the conductors. Note that (20) and (21) imply that the magnetic field distribution can be obtained from a solution of an electrostatic problem. It is clear that (20) and (21) in conjunction with (18) and (19) establish the spatial orthogonality of the electric and magnetic fields.

In terms of (20) and (21) the effective dielectric constant k_{eff}, defined as the ratio of the stored electric energy per unit length with and without the substrate present ($k = 1$) at a specified charge Q, is expressible in the form

$$k_{\text{eff}} = K(g, k). \tag{22}$$

In a similar fashion, the magnitude of the effective relative permeability μ_{eff} equal to the ratio of the stored magnetic energy per unit length, with and without the substrate present ($\mu = 1$) for a specified current is of the form

$$\frac{1}{\mu_{eff}} = K(g, \mu^{-1}) \tag{23}$$

where the energy density function K is given by

$$K(g, k) = \frac{\int_{\mathcal{A}} |\nabla F(x, y, 1)|^2 \, dx \, dy}{\int_{\mathcal{A}} k(x, y) \, |\nabla F(x, y, k)|^2 \, dx \, dy}. \tag{24}$$

Here $\mathcal{A}$ denotes the cross section of the propagating structure, excluding the conductors. The function $k(x, y)$ equals k in the substrate cross section, and unity in the air space above it. The parameter g is a geometrical factor, which equals w/h for the simple microstrip configuration of Fig. 1.

From (22) and (23) we obtain the interesting and useful duality relation

$$\mu_{eff}(g, \mu) = \frac{1}{k_{eff}(g, \mu^{-1})} \tag{25}$$

which was to be proven.

How realistic is the assumption of proportionality between $j(s)$ and $\sigma(s)$? For a system with a homogeneous cross section propagating a pure TEM mode, it is strictly correct. For a *non*homogeneous system, as we are considering here, $\sigma(s)$ and $j(s)$ *cannot* have identical distributions and this, *because* we postulate a TEM mode. Our reasoning is as follows. Suppose we have a TEM mode, and we assume $\sigma(s)$ and $j(s)$ to be proportional. Now consider a change in the dielectric constant of the substrate. Surely this will alter the charge distribution. By our assumption, the current distribution must also change, and so must the magnetic field distribution. But this cannot happen for a TEM mode, because the dielectric cannot affect the magnetic field.

Experience has shown that the magnetic field distribution, or more precisely, the inductance per unit length of microstrip, is *not* affected noticeably by the presence of a dielectric substrate. We can only conclude then that the charge and current distributions apparently do not deviate appreciably from proportionality and that the capacitance and inductance per unit length are not sensitive so much to the precise distribution of charge and current on the conductors, as they are to the geometrical configuration of the conductors and the substrate.

References

[1] H. A. Wheeler, "Transmission-line properties of parallel strips separated by a dielectric sheet," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-13, pp. 172–186, Mar. 1965.

[2] H. E. Stinehelfer, Sr., "An accurate calculation of uniform microstrip transmission lines," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-16, pp. 439–444, July 1968.

[3] E. Yamashita and R. Mittra, "Variational method for the analysis of microstrip lines," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-16, pp. 251–256, Apr. 1968.

[4] J. D. Welch and J. J. Pratt, "Losses in microstrip transmission systems for integrated microwave circuits," *NEREM Rec. 8*, pp. 100–101, 1966.

[5] R. A. Pucel, D. Massé, and C. P. Hartwig, "Losses in microstrip," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-16, pp. 342–350, June 1968; also "Correction to 'Losses in microstrip'," *IEEE Trans. Microwave Theory Tech.* (Corresp.), vol. MTT-16, p. 1064, Dec. 1968.

[6] T. Kaneki, "Analysis of linear microstrip using an arbitrary ferromagnetic substance as the substrate," *Electron. Lett.*, vol. 5, pp. 465–466, Sept. 18, 1969.

[7] H. B. Palmer, "The capacitance of a parallel-plate capacitor by the Schwarz–Christoffel transformation," *Elec. Eng.*, pp. 363–366, Mar. 1939.

[8] D. J. Massé and R. A. Pucel, "Microstrip propagation on magnetic substrates—Part II: Experiment," *IEEE Trans. Microwave Theory Tech.*, this issue, pp. 309–313.